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PROJECTIVE AND AFFINE GEOMETRY OF PATHS

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1. A path is defined as any curve given parametrically by a set of solutions of the differential equations

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \quad \Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha} \tag{1.1}$$

The same system of paths may, however, be defined in terms of the same system of coördinates by another set of differential equations

$$\frac{d^2x^i}{dt^2} + \Lambda^i_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0, \quad \Lambda^i_{\alpha\beta} = \Lambda^i_{\beta\alpha} \tag{1.2}$$

The functions $\Gamma^i_{\alpha\beta}$ give rise to one definition of covariant differentiation and infinitesimal parallelism, and the functions $\Lambda^i_{\alpha\beta}$ to a different definition of these operations and relations. But both types of parallelism and of covariant differentiation refer to the same system of paths.

The theorems which state general properties of a system of paths, without restriction as to the scope of the theorems, constitute a projective geometry of paths. The theorems which state properties of the paths and of a particular set of functions, $\Gamma^i_{\alpha\beta}$, i.e., of a particular definition of infinitesimal parallelism, constitute an affine geometry of paths.

The discovery that there can be more than one affine geometry for a given system of paths is due to H. Weyl who also obtained an important tensor which he calls the *projective curvature*. Weyl's results are published in the *Göttinger Nachricten* for 1921 (p. 99), which has only recently reached this country.

The question whether there could be two sets of differential equations such as (1.1) and (1.2) for the same set of paths was raised by Professor Eisenhart (cf. this volume, p. 233) and answered in the negative under the

assumption that "the parameter s is the same for all the paths." The following paragraphs will serve to show wherein this assumption restricts the problem and also provide a second method of approaching the projective geometry of paths.

2. Let us first see how the differential equations (1.1) and (1.2) can give rise to the same system of paths. Any solution of the differential equations (1.1) can be written in the form (cf. p. 193 of this volume)

$$x^i = q^i + \psi^i (\xi^1 s, \xi^2 s, \ldots, \xi^n s) \tag{2.1}$$

in which $q^1, q^2, \ldots, q^n$ are the coördinates of an arbitrary point and $\xi^1, \xi^2, \ldots, \xi^n$ an arbitrary set of values of $dx^1/ds, dx^2/ds, \ldots, dx^n/ds$ at this point. The solution of (1.2) representing the same paths as (2.1) may be written in the form

$$x^i = q^i + \mathrm{X}^i(\eta^1 t, \eta^2 t, \ldots, \eta^n t) \tag{2.2}$$

in which $\eta^1, \eta^2, \ldots, \eta^n$ is a set of values of $dx^1/dt, dx^2/dt, \ldots, dx^n/dt$ at the point q. By setting up a correspondence between the values of s and t which correspond to the same point of this path we define s as a function of t. This function depends on the point q and the direction of the path through q. Hence if $y^i = \eta^i t$, we may write

$$s = f (q^1, q^2, \ldots, q^n, y^1, y^2, \ldots, y^n) \tag{2.3}$$

for the value of s which corresponds to the same point as $q^1, q^2, \ldots, q^n$ and $y^1, y^2, \ldots, y^n$. As a transformation of the differential equation this may also be written.

$$s = f \left(x^1, x^2, \ldots, x^n, \frac{dx^1}{dt} t, \frac{dx^2}{dx} t, \ldots, \frac{dx^n}{dt} t\right) \tag{2.4}$$

Let us now multilpy (1.1) by $(\partial f/\partial t)^2$ and add a term to each side so as to obtain

$$\frac{d^2x^i}{ds^2}\left(\frac{\partial f}{\partial t}\right)^2 + \frac{dx^i}{ds}\frac{\partial^2 f}{\partial t^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\left(\frac{\partial f}{\partial t}\right)^2 = \frac{dx^i}{ds}\frac{\partial^2 f}{\partial t^2} \tag{2.5}$$

which is the same as

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = \frac{dx^i}{dt}\Theta\left(x^1, x^2, \ldots, x^n, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \ldots, \frac{dx^n}{dt}, t\right) \tag{2.6}$$

a differential equation in which t has exactly the same significance as in (1.2).

3. Let us now subtract (1.2) from (2.6). The result is

$$\varphi^i_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = \Theta\frac{dx^i}{dt} \tag{3.1}$$

in which we are defining the functions $\varphi^{i}_{\alpha\beta}$ by the equations

$$\varphi^{i}_{\alpha\beta} = \Gamma^{i}_{\alpha\beta} - \Lambda^{i}_{\alpha\beta} \tag{3.2}$$

The functions thus defined are the components of a tensor which is contravariant with respect to i and covariant with respect to α and β. This is because a transformation to new independent variables $z^1, z^2, \ldots, z^n$, changes the Γ's according to the formula

$$\Gamma^{i}_{\alpha\beta}(z) = \Gamma^{j}_{\gamma\delta}(x) \frac{\partial x^\gamma}{\partial z^\alpha} \frac{\partial x^\delta}{\partial x^\beta} \frac{\partial z^i}{\partial x^j} + \frac{\partial^2 x^j}{\partial z^\alpha \partial z^\beta} \frac{\partial z^i}{\partial x^j} \tag{3.3}$$

This tensor defines a covariant vector by means of the formula

$$\varphi_\alpha = \frac{1}{n+1} \varphi^{i}_{i\alpha} \tag{3.4}$$

By eliminating Θ from the equations (3.1) we obtain

$$\left(\varphi^{i}_{\alpha\beta} \frac{dx^j}{dt} - \varphi^{j}_{\alpha\beta} \frac{dx^i}{dt}\right) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0 \tag{3.5}$$

Remembering that the values of $dx^1/dt, dx^2/dt, \ldots, dx^n/dt$ are arbitrary it is easy to infer from (3.5) that

$$\varphi^{i}_{\alpha\beta} = \delta^{i}_{\alpha} \varphi_\beta + \delta^{i}_{\beta} \varphi_\alpha \tag{3.6}$$

where δ^i_α is 1 or 0 according as $i = \alpha$ or $i \neq \alpha$.

When (3.6) is substituted in (3.1) there results

$$2 \varphi_\gamma \frac{dx^\gamma}{dt} = \Theta. \tag{3.7}$$

4. Conversely it can be shown that if we start with the differential equation (1.1) and any covariant vector φ_α, we can find a second differential equation of the type (1.2) which defines the same set of paths as (1.1). The first step is to define the functions $\varphi^{i}_{\alpha\beta}$ by means of equations (3.6). Since δ^i_α is a tensor and φ_α is a covariant vector it follows that $\varphi^{i}_{\alpha\beta}$ is a tensor of the third order. We then define $\Lambda^{i}_{\alpha\beta}$ by means of the equations (3.2) and write the differential equations (1.2) which are by definition identical with

$$\frac{d^2x^i}{dt^2} + \Gamma^{i}_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 2 \varphi_\alpha \frac{dx^\alpha}{dt} \frac{dx^i}{dt} \tag{4.1}$$

If we compare (4.1) with (2.5) we see that (4.1) is obtainable from (1. 1) by means of the substitution

$$s = e^{2\varphi_\gamma \frac{dx^\gamma}{dt}} + h(x) \tag{4.2}$$

where $h(x)$ is an arbitrary function of $x^1, x^2, \ldots, x^n$.

It has been proved that: (1) If (1.1) and (1.2) are differential equations which define the same system of paths then the functions $\Gamma^{\,i}_{\alpha\beta}$ and $\Lambda^{\,i}_{\alpha\beta}$ are related by the equations

$$\Gamma^{\,i}_{\alpha\beta} - \Lambda^{\,i}_{\alpha\beta} = \varphi^{\,i}_{\alpha\beta} = \delta^{\,i}_{\alpha}\varphi_{\beta} + \delta^{\,i}_{\beta}\varphi_{\alpha} \tag{4.3}$$

in which φ_α is a vector. (2) If (1.1) are the differential equations of a system of paths and φ_α is any covariant vector then there exists a second set of differential equations (1.2) for the same system of paths, the functions $\Lambda^{\,i}_{\alpha\beta}$ being defined by (4.3).

The formula (3.7) reduces to that obtained by Eisenhart on page 235 of this volume in the case when s is a function of t alone. This makes it evident that his work must be taken as referring to a geometry in which there is an element of distance ds defined by means of functions of position. In this sense his theorem on page 235 will be found not to be in contradiction with the theorem (2) above.

A DIRECT METHOD OF TESTING COLOR VISION IN LOWER ANIMALS

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1. When spectral lights of different wave-lengths are directed on the photoreceptors of lower organisms, their reactions show that one of the two lights is, physiologically, the more effective. Even when the unequal distribution of energy in the spectrum is equated for, there still remains a difference in the physiological effectiveness of the stimuli. (Laurens and Hooker.[1]) This difference must be dependent upon wave-length. But it is also evident to the color blind eye (Koenig[2]). We cannot therefore argue that the differential response on the part of the organism is proof of color division.

Researches which have approached the question of color vision in lower organisms by means of *conditioned behavior* have not been done with pure spectral lights. Even when pure spectral lights are used, the difference in brightness must be equalized, not, however, in accordance with the luminosity curve of the human eye, but with that of the organisms whose color vision is to be tested. Obviously it is difficult to obtain a luminosity curve free from extraneous factors from the reactions of organisms sufficiently complex and intelligent to exhibit conditioned behavior.